Optimal Behavioral Matching

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Abstract

The paper studies efficient matchings in the context in which payoffs depend on the actions of the group members, instead of on their characteristics. To describe the payoff maximizing matching we re-express the model in terms of behavioral types and introduce the idea of behavioral matching. The analysis relies on an association order that has just been recently used in economics, namely, the increasing and supermodular order. As a by-product of the efficient characterization, we show that positive assortative matching is payoff maximizing in the case of increasing and supermodular payoffs if (and only if) the group members coordinate in the maximum equilibrium.

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1 Introduction

The classical theory of assortative matching shows that when payoffs display complementarities in some characteristic of people then positive assortative matching often emerges as an efficient outcome. To fix ideas, let us consider a classroom that has Girls and Boys. The students are either high- or low-ability levels. The teacher wants to assign students to groups of two (one Girl and one Boy) to work in a specific project. If the quality of the project displays complementarities in ability levels, then the aggregate quality of projects across all pairs of students is often maximized when the groups are formed with students of the same ability level. This is referred in the economic literature as positive assortative matching. This paper was started by the recognition that (quite often) the payoffs of the matchings depend on the *actions* of people instead of on their characteristics. In this alternative set-up we describe the set of efficient matchings and study whether they are assortative.

Let us consider again the example of the classroom. In this application one argue that the quality of the project depends on the effort level exerted by the students in each group. In this case, the ability might be just a determinant of the effort that the student selects. The effort level of the student might also depend on the effort selected by the other member of the group. The distinctive element in our set-up is that while the characteristics of people are exogenous to the model and not affected by the matching, the actions of the group members endogenously emerge as the outcome of a game. Moreover, the induced game might have multiple equilibria. Thus, to characterize the payoff maximizing matchings we need to accommodate endogeneity of choices and multiplicity of equilibria.

Specifically, we consider a population composed of two groups of people. The principal forms pairs of people with one member of each group. Each pair of people plays a binary game of strategic complements. We capture complementarities by assuming that payoffs display the single-crossing property between own action and the action of the other group member. In the case of the classroom, this implies that the effort level of one of the group members increases with the effort level selected by the other one. The choices in each pair of people emerge as a Nash equilibrium of the induced game. The principal cares about the choices of each pair via a payoff function. The initial aim of the paper is to characterize the set of matchings that maximize aggregate payoffs for payoff functions that are increasing and supermodular.

To achieve this goal, we first map the distribution of preferences (or utility functions) of people in each group into a distribution of implied behavior. We describe a behavioral type in terms of the action that a given person would choose as a function of the action selected by the other group member. The assumption of utility maximization means that each possible behavioral type corresponds to a best-reply function. Under the single-crossing property, the set of consistent behavioral types reduces to the set of best-replies that are increasing. We then notice that each pair in the population can be thought of as a pair of behavioral types. This approach allows us to define a *behavioral matching* as a bivariate distribution on the set of behavioral types. To each pair of behavioral types corresponds an equilibrium set. For the pairs that generate multiple equilibria, we assume that there is an arbitrary equilibrium selection rule. Altogether, each behavioral matching generates a distribution of choices that can be used to construct the aggregate payoffs for the principal. Under this behavioral set-up, we take advantage of the increasing and supermodular stochastic order to characterize the set of behavioral matchings that maximize the aggregate payoffs.

As a by-product, the conditions we impose in preferences induce a natural order on the set of behavioral types. This allows us to explore whether positive assortative matching is payoff maximizing in our model. We find that the answer is yes if, and only if, people coordinate on the highest equilibrium. For any other equilibrium selection rule, the payoff maximizing matching avoids forming pairs that generate multiple equilibria.

We now connect our work with the existing literature. The concept of positive assortative matching was introduced by Becker (1973, 1974) in his extremely influential work on the marriage market. There is a vast literature in economics following up on this idea. As we just described, we study a similar matching problem but for the case in which payoffs depend on actions (not on characteristics). Durlauf and Seshadri (2003) consider the possibility that choices enter payoffs but do not formally deal with multiplicity of equilibria. The idea of defining behavioral types in terms of best-replies has been used by Lazzati, Quah, and Shirai (2021) to test Nash equilibrium behavior in supermodular games. We extend the approach to groups and introduce the idea of *behavioral matching*. The paper relies on the increasing and supermodular stochastic order. The supermodular stochastic order has been recently used in economics (for very different purposes) by Amir and Lazzati (2016), Athey and Levin (2001), and Levin (2001), among others. Dziewulski and Quah (2014) use the increasing and supermodular order to derive a revealed preference test for production with complementarities, and Lazzati (2020) uses this order to study the co-diffusion process of complementary technologies. Meyer and Strulovici (2012, 2015) provide a useful characterization of these association orders coupled with many interesting applications to economics.

The rest of the paper is organized as follows. Section 2 presents the model and the objective of our analysis. Section 3 introduces the concept of behavioral matching and characterizes the one that maximizes aggregate payoffs. Section 4 assumes the principal observes the distribution of choices for some initial matching. This section shows that the principal can use the observed choices to learn the type of each person in the population via a simple re-matching. This information could be used to implement the efficient matching. Section 5 concludes. The proofs are collected in Section 6.

2 The Model

The population is composed of two groups of people, G_1 and G_2. For example, G_1 might be girls and G_2 might be boys. The principal wants to arrange the population into pairs of people with one member of each group in each pair. We will refer to Player 1 as a person from Group 1 and Player 2 as a person from Group 2. We assume the two groups have the same size. That is, $|\mathrm{G}_1| = |\mathrm{G}_2|$.

Each pair of people plays a binary game. The action space of each player is $\{0,1\}$. For example, 0 might be low effort in a group project and 1 might indicate high effort. There is a distribution of preferences in each group, allowing for arbitrary heterogeneity across people. Each pair of people is associated to a realization of the random utilities described as follows

$$\mathrm{U}_1(x_1,x_2) : \{0,1\}\times\{0,1\} \to \mathbb{R} \text{ and } \mathrm{U}_2(x_2,x_1) : \{0,1\}\times\{0,1\} \to \mathbb{R}$$

where x_1 and x_2 are the actions selected by Players 1 and 2, respectively. Let P_1 and P_2 be the distributions of preferences in Groups 1 and 2, respectively. Each pair of marginals induces a set of joint distributions P_{12}. Consistency requires

$$\int \mathrm{P}_{12}\mathrm{dP}_2 = \mathrm{P}_1 \text{ and } \int \mathrm{P}_{12}\mathrm{dP}_1 = \mathrm{P}_2.$$

The distribution of preferences might differ in the two groups. We are interested in distributions of preferences with support on utility functions that have unique maximizers and satisfy the single-crossing property.

Definition (Single-Crossing Property) $\mathrm{U}_g(x_g, x_{-g})$ has the single-crossing property in x_g and x_{-g} if, for $g = 1,2$,

$$\mathrm{U}_g(1,0) - \mathrm{U}_g(0,0) \geq (>)\, 0 \Longrightarrow \mathrm{U}_g(1,1) - \mathrm{U}_g(0,1) \geq (>)\, 0.$$

Along the analysis will use a simple example to illustrate our ideas.

Example: Let us define $\mathrm{U}_g(x_g, x_{-g})$ as the realization of a random payoff

$$\mathrm{U}_g(1,0,a) = (a-50)x_g + 10x_g x_{-g} \text{ for } g = 1,2$$

where a is uniformly distributed in the interval $[0,100]$. For instance, if the actions of the players indicate effort levels, then a might reflect the ability of the person. It can be easily checked that this utility function has the single-crossing property in x_g and x_{-g}. ■

As we initially stated, each pair of people plays a binary game. We use Nash equilibrium in pure strategies as our solution concept. It is well-known that when the utilities satisfy the single-crossing property, then the equilibrium set is non-empty (see, e.g., Milgrom and Shannon (1994)). It has also been established that multiple equilibria cannot be ruled out in this class of games. Let $\mathrm{NE}(\mathrm{U}_1,\mathrm{U}_2) \subseteq \{0,1\} \times \{0,1\}$ be the equilibrium set induced by $\mathrm{U}_1,\mathrm{U}_2$. Let $\lambda(\cdot \mid \mathrm{U}_1,\mathrm{U}_2)$ be an equilibrium selection rule, where $\lambda(x_1,x_2 \mid \mathrm{U}_1,\mathrm{U}_2)$ is the fraction of groups with preferences $\mathrm{U}_1,\mathrm{U}_2$ that select the pair of choices x_1,x_2. We assume $\lambda(x_1,x_2 \mid \mathrm{U}_1,\mathrm{U}_2) = 0$ for $x_1,x_2 \notin \mathrm{NE}(\mathrm{U}_1,\mathrm{U}_2)$ and $\sum_{x_1,x_2\in\{0,1\}\times\{0,1\}} \lambda(x_1,x_2 \mid \mathrm{U}_1,\mathrm{U}_2) = 1$.

The principal cares about the choices of each pair of people via a payoff function $\pi : \{0,1\} \times \{0,1\} \to \mathbb{R}$. The aggregate payoff for the principal is given by

$$\int \sum_{x_1,x_2\in\{0,1\}\times\{0,1\}} \pi(x_1,x_2)\,\lambda(x_1,x_2 \mid \mathrm{U}_1,\mathrm{U}_2)\,d\mathrm{P}_{12}.$$

We want to characterize the set of joint distributions P_{12} that maximize the aggregate payoffs for each pair of marginals (P_1 and P_2) and certain types of payoff functions. Specifically, we are interested in payoffs that are increasing and supermodular.

Definition (Properties of the Payoff Function) The payoff function π is increasing if

$$\pi(1,1) \geq \pi(0,1) \parallel \pi(1,0) \geq \pi(0,0).$$

It is supermodular if

$$\pi(1,1) + \pi(0,0) \geq \pi(0,1) + \pi(1,0).$$

In the case of increasing payoffs, the principal prefers high over low actions. Supermodular payoffs capture the idea of complementarities. These two properties are quite natural in many applications we are interested in. Supermodularity also plays a key role in the equilibrium and efficiency proofs of assortative matching in various models.

Example: Let us consider the following payoff function for the principal

$$\pi(x_1,x_2) = 1(x_1 = 1) + 1(x_2 = 1) + 1(x_1 = 1)\,1(x_2 = 1)$$

where $1(\cdot)$ is the standard indicator function. Note that π is increasing and supermodular.

3 Optimal Behavioral Matching

In this section, we characterize the distributions of groups that maximize aggregate payoffs for payoff functions that are increasing and supermodular. Since the principal cares about the actions selected by each pair of people, it is convenient to express the model in terms of behavioral types. The notion of behavioral types will allow us to think about a distribution of groups as a behavioral matching. We start by re-writing the model in this way, and then characterize the optimal behavioral matching.

3.1 Behavioral Matching

A **behavioral type** describes the action that a person would choose as a function of the action selected by the other player. According to this definition, each person might belong to one of four possible behavioral types.

Other Player Action		**Behavioral Type**
0	**1**	
0	0	Type 1 (T_1)
0	1	Type 2 (T_2)
1	1	Type 3 (T_3)
1	0	Type 4 (T_4)

For instance, a Type 2 player (T_2) selects action 0 if the other player selects action 0 and action 1 if the other player selects 1. The other three types can be similarly interpreted. Let us indicate by $Q_1 = \left(Q_1^1, Q_2^1, Q_3^1, Q_4^1\right)$ and $Q_2 = \left(Q_1^2, Q_2^2, Q_3^2, Q_4^2\right)$ the distributions of types in Groups 1 and 2, respectively. Naturally, for $g = 1, 2$, we have that

$$Q_1^g, Q_2^g, Q_3^g, Q_4^g \geq 0 \quad \text{and} \quad Q_1^g + Q_2^g + Q_3^g + Q_4^g = 1.$$

When players are utility-maximizers and their maximizers are singletons, then each behavioral type corresponds to a best-reply function. In this case, the random utilities can be decomposed into the four behavioral types we just described. If, in addition, the utility functions satisfy the single-crossing property, then the best-replies are increasing. In terms of behavioral types, this means that $\mathrm{Q}_4^g = 0$ for $g = 1, 2$. That is, only players of Types 1, 2, and 3 are consistent with the hypothesis of maximizing single-valued utilities that have the single-crossing property. Altogether, each marginal distribution P_g induces a distribution of behavioral types, for $g = 1, 2$,

$$\begin{aligned}
\mathrm{Q}_1^g &= \mathrm{E}_{\mathrm{P}_g}\left[1\left(\mathrm{U}_g(0,0) > \mathrm{U}_g(1,0) \wedge \mathrm{U}_g(0,1) > \mathrm{U}_g(1,1)\right)\right] \\
\mathrm{Q}_2^g &= \mathrm{E}_{\mathrm{P}_g}\left[1\left(\mathrm{U}_g(0,0) > \mathrm{U}_g(1,0) \wedge \mathrm{U}_g(1,1) > \mathrm{U}_g(0,1)\right)\right] \\
\mathrm{Q}_3^g &= \mathrm{E}_{\mathrm{P}_g}\left[1\left(\mathrm{U}_g(1,0) > \mathrm{U}_g(0,0) \wedge \mathrm{U}_g(1,1) > \mathrm{U}_g(0,1)\right)\right]
\end{aligned} .$$

We take the distributions of behavioral types as given.

Example: Recall that, for each group, the utility function is

$$\mathrm{U}_g(1, 0, a) = (a - 50)x_g + 10x_g x_{-g}$$

with a being uniformly distributed in $[0, 100]$. It follows that

$$\mathrm{Q}_1^g = 40/100,\ \mathrm{Q}_2^g = 10/100 \text{ and } \mathrm{Q}_3^g = 50/100 \text{ for } g = 1, 2.$$

Note that $\mathrm{Q}_4^1 = \mathrm{Q}_4^2 = 0$. Since the utilities satisfy the single-crossing condition, this is consistent with our previous statement. ■

We stated earlier that each pair of people plays a binary game and we use Nash equilibrium in pure strategies as our solution concept. Thus, we can associate each pair of possible types with an equilibrium set. The next table captures this idea.

Pair of Behavioral Types	**Equilibrium Set**
$\mathrm{T}_1\mathrm{T}_1, \mathrm{T}_1\mathrm{T}_2, \mathrm{T}_2\mathrm{T}_1$	$0, 0$
$\mathrm{T}_1\mathrm{T}_3$	$0, 1$
$\mathrm{T}_3\mathrm{T}_1$	$1, 0$
$\mathrm{T}_2\mathrm{T}_3, \mathrm{T}_3\mathrm{T}_2, \mathrm{T}_3\mathrm{T}_3$	$1, 1$
$\mathrm{T}_2\mathrm{T}_2$	$0, 0$ and $1, 1$

.

Multiple equilibria occur when a pair is formed with two people of Type 2. In this case, we assume $1,1$ is selected with probability α, and $0,0$ is chosen with probability $1-\alpha$.

Let $\mathrm{M} = (\mathrm{M}_{st})_{s,t=1,2,3}$ be a distribution of pairs of people, where M_{st} is the fraction of pairs with a Type s person as Player 1 and a Type t person as Player 2. We will refer to each M as a **behavioral matching**. The distribution of behavioral types imposes some restrictions on M. Specifically,

$$\sum\nolimits_{t=1,2,3} \mathrm{M}_{st} = \mathrm{Q}_s^1 \text{ for } s = 1,2,3 \text{ and } \sum\nolimits_{s=1,2,3} \mathrm{M}_{st} = \mathrm{Q}_t^2 \text{ for } t = 1,2,3.$$

We indicate by $\mathcal{M}$ the set of all possible behavioral matchings.

Each behavioral matching M induces a distribution of equilibrium choices $\mathrm{P}(\cdot \mid \mathrm{M}, \alpha)$

$$\begin{aligned}
\mathrm{P}(0,0 \mid \mathrm{M}, \alpha) &= \mathrm{M}_{11} + \mathrm{M}_{12} + \mathrm{M}_{21} + (1-\alpha)\,\mathrm{M}_{22} \\
\mathrm{P}(0,1 \mid \mathrm{M}, \alpha) &= \mathrm{M}_{13} \\
\mathrm{P}(1,0 \mid \mathrm{M}, \alpha) &= \mathrm{M}_{31} \\
\mathrm{P}(1,1 \mid \mathrm{M}, \alpha) &= \alpha \mathrm{M}_{22} + \mathrm{M}_{23} + \mathrm{M}_{32} + \mathrm{M}_{33}
\end{aligned}.$$

In terms of the initial model, we have that

$$\mathrm{P}(x_1, x_2 \mid \mathrm{M}, \alpha) = \mathrm{E}_{\mathrm{P}_{12}}\left[\lambda\left(x_1, x_2 \mid \mathrm{U}_1, \mathrm{U}_2\right)\right] \text{ for each } x_1, x_2 \in \{0,1\} \times \{0,1\}.$$

Recall that the principal cares about the choices of each pair of people via a payoff function $\pi : \{0,1\}^2 \to \mathbb{R}$. The aggregate payoff for the principal can be re-written as follows

$$\Pi(\mathrm{M}, \alpha) = \pi(0,0)\,\mathrm{P}(0,0 \mid \mathrm{M}, \alpha) + \pi(0,1)\,\mathrm{P}(0,1 \mid \mathrm{M}, \alpha) + \pi(1,0)\,\mathrm{P}(1,0 \mid \mathrm{M}, \alpha) + \pi(1,1)\,\mathrm{P}(1,1 \mid \mathrm{M}, \alpha).$$

Under this specification, we want to characterize the set of behavioral matchings that maximize the total payoff

$$\max\nolimits_{\mathrm{M}} \left\{\Pi(\mathrm{M}, \alpha) : \mathrm{M} \in \mathcal{M} \text{ and } \alpha \in [0,1]\right\}$$

for payoff functions π that are increasing and supermodular (ISPM). We will refer to this sub-set as $\mathcal{M}^{\mathrm{ISPM}}$.

Let us use our example again to illustrate the new concepts.

Example: The payoff function for the principal is

$$\pi(x_1, x_2) = 1(x_1 = 1) + 1(x_2 = 1) + 1(x_1 = 1)\, 1(x_2 = 1).$$

We obtained earlier that

$$\mathrm{Q}_1^g = 40/100,\ \mathrm{Q}_2^g = 10/100 \text{ and } \mathrm{Q}_3^g = 50/100 \text{ for } g = 1, 2.$$

Thus, for each behavioral matching M, we have that

$$\Pi(\mathrm{M}, \alpha) = \mathrm{M}_{13} + \mathrm{M}_{31} + 3(\alpha \mathrm{M}_{22} + \mathrm{M}_{23} + \mathrm{M}_{32} + \mathrm{M}_{33}).$$

The purpose of the principal is to solve the next problem

$$\max_{\mathrm{M}} \{\Pi(\mathrm{M}, \alpha) = \mathrm{M}_{13} + \mathrm{M}_{31} + 3(\alpha \mathrm{M}_{22} + \mathrm{M}_{23} + \mathrm{M}_{32} + \mathrm{M}_{33}) : \mathrm{M} \in \mathcal{M}\}.$$

In this case, any $\mathrm{M} \in \mathcal{M}$ satisfies

$$\sum\nolimits_t \mathrm{M}_{1t} = \sum\nolimits_s \mathrm{M}_{s1} = \frac{10}{100},\ \sum\nolimits_t \mathrm{M}_{2t} = \sum\nolimits_s \mathrm{M}_{s2} = \frac{40}{100}, \text{ and } \sum\nolimits_t \mathrm{M}_{3t} = \sum\nolimits_s \mathrm{M}_{s3} = \frac{50}{100}.$$

This set includes all possible behavioral matchings that can be attained with the given distribution of behavioral types in each group. ■

3.2 Payoff Maximizing Behavioral Matchings

This section characterizes the sub-set of behavioral matchings $\mathcal{M}^{\mathrm{ISPM}} \subseteq \mathcal{M}$ that maximize aggregate payoffs $\Pi(\mathrm{M}, \alpha)$ for any increasing and supermodular payoff function π.

Let us formally say that $\mathrm{M} \in \mathcal{M}^{\mathrm{ISPM}}$ if, for any increasing and supermodular payoff function π, we have that

$$\Pi(\mathrm{M}, \alpha) \geq \Pi(\mathrm{M}\prime, \alpha) \text{ for all } \mathrm{M}\prime \in \mathcal{M} \backslash \mathcal{M}^{\mathrm{ISPM}} \text{ and all } \alpha \in [0, 1].$$

Recall that the aggregate payoff for the principal can be written as follows

$$\Pi(\mathrm{M}, \alpha) = \pi(0,0)\,\mathrm{P}(0,0 \mid \mathrm{M}, \alpha) + \pi(0,1)\,\mathrm{P}(0,1 \mid \mathrm{M}, \alpha) + \pi(1,0)\,\mathrm{P}(1,0 \mid \mathrm{M}, \alpha) + \pi(1,1)\,\mathrm{P}(1,1 \mid \mathrm{M}, \alpha).$$

Notice that Π can be described as the expected value of π with respect to the distribution of choices induced by the behavioral matching M. The characterization of $\mathcal{M}^{\text{ISPM}}$ relies on the increasing and supermodular stochastic (ISPM) order. We define this order next, and subsequently establish its relevance in our problem.

Definition (ISPM Order) Let M,M$\prime \in \mathcal{M}$. We say $\mathrm{P}(\cdot \mid \mathrm{M}, \alpha) \geq_{\text{ISPM}} \mathrm{P}(\cdot \mid \mathrm{M}\prime, \alpha)$ if

$$\begin{aligned}
\mathrm{P}(1,0 \mid \mathrm{M}\prime,\alpha) + \mathrm{P}(1,1 \mid \mathrm{M}\prime,\alpha) &\geq \mathrm{P}(1,0 \mid \mathrm{M},\alpha) + \mathrm{P}(1,1 \mid \mathrm{M},\alpha) \\
\mathrm{P}(0,1 \mid \mathrm{M}\prime,\alpha) + \mathrm{P}(1,1 \mid \mathrm{M}\prime,\alpha) &\geq \mathrm{P}(0,1 \mid \mathrm{M},\alpha) + \mathrm{P}(1,1 \mid \mathrm{M},\alpha) \\
\mathrm{P}(1,1 \mid \mathrm{M}\prime,\alpha) &\geq \mathrm{P}(1,1 \mid \mathrm{M},\alpha).
\end{aligned}$$

In the case of bivariate Bernoulli distributions, the ISPM stochastic order coincides with the upper orthant order. (This equivalence has been established by Scarsini (1998).) According to the previous definition, a probability distribution is larger than another one if it has higher probabilities for all upper orthant sets. In our model, this is the same as to say that the larger distribution has a higher proportion of Players 1 selecting action 1, Players 2 selecting action 1, and pairs of Players 1 and 2 selecting the pair of actions $1, 1$.

The next result shows that this order can be used to compare the expected payoffs of functions that are increasing and supermodular.

Theorem (ISPM Order) $\Pi(\mathrm{M}, \alpha) \geq \Pi(\mathrm{M}\prime, \alpha)$ for any increasing and supermodular π if, and only if,

$$P(\cdot \mid \mathrm{M}, \alpha) \geq_{\text{ISPM}} P(\cdot \mid \mathrm{M}\prime, \alpha).$$

We finally apply these results to characterize the optimal behavioral matching.

Proposition 1 *$\mathcal{M}^{ISPM}$ is characterized by*

$$M_{11} = \min\left\{Q_1^1, Q_1^2\right\}, \ M_{23} = \min\left\{Q_2^1, Q_3^2\right\}, \ \textit{and } M_{32} = \min\left\{Q_3^1, Q_2^2\right\}.$$

Moreover, it is a singleton.

Let us make a few remarks about the last result. First, notice that the purpose of the maximization problem was to find the set of *all* behavioral matchings that are payoff maximizing for *any* increasing and supermodular profit function. The fact that the optimal behavioral matching is unique is a strong result. Second, let us say that a best-reply function is larger than another one if the person is more willing to select action 1. This idea allows us to rank the behavioral types as follows

$$\mathrm{T}_1 < \mathrm{T}_2 < \mathrm{T}_3.$$

Using this order, it is readily verified that the optimal matching is not assortative. That is, it does not pair people in an increasing way. Indeed, the matching avoids forming pairs of types $\mathrm{T}_2\mathrm{T}_2$. Recall that this pair of behavioral types select actions $0, 0$ with probability $1 - \alpha$ and actions $1, 1$ with probability α. Instead, the optimal matching pairs Type 2 people with people of Type 3. In doing so, the matching induces actions $1, 1$ with probability 1. That is, the optimal matching pairs types to shift actions up in the area of multiple equilibria.

Example: In our initial example, the optimal behavioral matching is given by

$$\mathrm{M}_{11} = 40/100,\ \mathrm{M}_{23} = \mathrm{M}_{32} = 10/100,\ \text{and}\ \mathrm{M}_{33} = 40/100.$$

Under this matching the aggregate payoff of the principal is

$$\Pi\left(\mathrm{M}, \alpha\right) = 180/100.$$

Notice that this behavioral matching generates the same average payoff irrespective of the equilibrium selection rule α. ■

We finally show that assortative matching is payoff maximizing if (and only if) players coordinate in the highest equilibrium, i.e. $\alpha = 1$.

Proposition 2 *Let us define the assortative matching $\mathcal{M}^A$ as follows*

$$M_{11} = Q_1^g,\ M_{22} = Q_2^g \text{ and } M_{33} = Q_3^g \text{ for } g = 1, 2.$$

Assume the distribution of types coincides in the two groups and $\alpha = 1$. Then, $\mathcal{M}^A \in \mathcal{M}^{ISPM}$.

Example: In our simple example, the assortative matching is given by

$$\mathrm{M}_{11} = 40/100,\ \mathrm{M}_{22} = 10/100,\ \text{and}\ \mathrm{M}_{33} = 50/100.$$

Under this matching the aggregate payoff of the principal is

$$\Pi\left(\mathrm{M}, \alpha\right) = 3\left(\alpha 10/100 + 50/100\right).$$

This payoff is lower than the one that corresponds to the optimal behavioral matching, 180/100, for any $\alpha < 1$. The two payoffs coincide when $\alpha = 1$. ■

4 Identification of Types

In the previous section we characterized the optimal behavioral matching. Let us now assume that the principal observes the choices for all pairs of people in the population and would like to infer the types of the players. This exercise is useful if the principal has the possibility of re-matching to increase the aggregate payoffs.

Notice that, without any extra assumption, the principal can recover the types for pairs that selected either $(0,1)$ or $(1,0)$. This follows from the fact that

$$\begin{array}{l} \mathrm{P}(0,1 \mid \mathrm{M}, \alpha) = \mathrm{M}_{13} \\ \mathrm{P}\left(1,0 \mid \mathrm{M}, \alpha\right) = \mathrm{M}_{31} \end{array}.$$

The other two pairs of choices can be generated by different pairs of behavioral types and their frequency also depends on the equilibrium selection rule. Specifically, we have that

$$\begin{array}{l} \mathrm{P}\left(0,0 \mid \mathrm{M}, \alpha\right) = \mathrm{M}_{11} + \mathrm{M}_{12} + \mathrm{M}_{21} + \left(1-\alpha\right)\mathrm{M}_{22} \\ \mathrm{P}\left(1,1 \mid \mathrm{M}, \alpha\right) = \alpha \mathrm{M}_{22} + \mathrm{M}_{23} + \mathrm{M}_{32} + \mathrm{M}_{33} \end{array}.$$

It follows that, without assuming any equilibrium selection rule, we can just partially identify pairs of types from the observed choices. The next table summarizes the information we have.

Observed Choices	**Partially Identified Set of Pairs of Behavioral Types**
$0, 0$	$\mathrm{T}_1\mathrm{T}_1, \mathrm{T}_1\mathrm{T}_2, \mathrm{T}_2\mathrm{T}_1, \mathrm{T}_2\mathrm{T}_2$
$0, 1$	$\mathrm{T}_1\mathrm{T}_3$
$1, 0$	$\mathrm{T}_3\mathrm{T}_1$
$1, 1$	$\mathrm{T}_2\mathrm{T}_3, \mathrm{T}_3\mathrm{T}_2, \mathrm{T}_3\mathrm{T}_3, \mathrm{T}_2\mathrm{T}_2$

But notice that if $\mathrm{P}(0, 1 \mid \mathrm{M}, \alpha)$ and $\mathrm{P}(1, 0 \mid \mathrm{M}, \alpha)$ are different from 0, the principal can eventually learn all types via re-matching. Specifically, the principal can use the T_1 and T_3 players as instruments to recover the type of the other players in each group.

To illustrate the previous claim, take a Player 2 that selected action 1 in a pair that selected $(0, 1)$. We know that this Player 2 is T_3. Then take Player 1 in a pair that selected $(0, 0)$. Match these two players. If Player 1 selects 0, then the person is T_1; if Player 1 selects 1, then the person is T_2. Similarly, take a Player 2 that selected 0 in a pair that selected $(1, 0)$. We know this Player 2 is T_1. Then take Player 1 in a pair that selected $(1, 1)$. Match these two players. If Player 1 selects 1, then the person is T_3; if the Player 1 selects 0, then the person is T_2. This procedure would allow the principal to learn the types of Group 1. A similar idea can be used to learn the types of Group 2.

5 Final Remarks

The paper studied optimal matching in a context in which payoffs depend on the actions of the group members, as compared to their characteristics. To describe the optimal matching we re-expressed the model in terms of behavioral types and introduced the idea of behavioral matching. As a by-product of the analysis, we showed that positive assortative matching is optimal in the case of increasing and supermodular payoffs if (and only if) players coordinate on the maximum equilibrium. The analysis relied on the increasing and supermodular stochastic order which have been recently used in other applications in economics.

We finally considered a situation in which the principal observes choices and is interesting in recovering types. This information would be relevant if he had the possibility of re-matching. We showed that while the distribution of types is just set identified from the observed choices, via a simple re-matching of players across groups, the principal could recover the types of the different players.

6 Proofs

Proof of Proposition 1. Notice that

$$\begin{aligned} \mathrm{P}(1,0 \mid \mathrm{M}) + \mathrm{P}(1,1 \mid \mathrm{M}) &= \mathrm{Q}_3^1 + \alpha \mathrm{M}_{22} + \mathrm{M}_{23} \\ \mathrm{P}(0,1 \mid \mathrm{M}) + \mathrm{P}(1,1 \mid \mathrm{M}) &= \mathrm{Q}_3^2 + \alpha \mathrm{M}_{22} + \mathrm{M}_{32} \\ \mathrm{P}(1,1 \mid \mathrm{M}) &= \alpha \mathrm{M}_{22} + \mathrm{M}_{23} + \mathrm{M}_{32} + \mathrm{M}_{33} \end{aligned}$$

Let M be such that $\mathrm{M}_{11} < \min\left\{\mathrm{Q}_1^1, \mathrm{Q}_1^2\right\}$. Thus, there are at least two pairs in M that belong to one of the next four categories

$$\mathrm{T}_1\mathrm{T}_2 \text{ and } \mathrm{T}_2\mathrm{T}_1 \quad \mathrm{T}_1\mathrm{T}_2 \text{ and } \mathrm{T}_3\mathrm{T}_1 \quad \mathrm{T}_1\mathrm{T}_3 \text{ and } \mathrm{T}_2\mathrm{T}_1 \quad \mathrm{T}_1\mathrm{T}_3 \text{ and } \mathrm{T}_3\mathrm{T}_1 \ .$$

Construct M′ by making one of the next rearrangements

$$\mathrm{T}_1\mathrm{T}_1 \text{ and } \mathrm{T}_2\mathrm{T}_2 \quad \mathrm{T}_1\mathrm{T}_1 \text{ and } \mathrm{T}_3\mathrm{T}_2 \quad \mathrm{T}_1\mathrm{T}_1 \text{ and } \mathrm{T}_2\mathrm{T}_3 \quad \mathrm{T}_1\mathrm{T}_1 \text{ and } \mathrm{T}_3\mathrm{T}_3 \ .$$

Note that, $\mathrm{P}(\cdot \mid \mathrm{M}\prime) \geq_{\mathrm{ISPM}} \mathrm{P}(\cdot \mid \mathrm{M})$. Since, $\mathrm{M}_{11} > \min\left\{\mathrm{Q}_1^1, \mathrm{Q}_1^2\right\}$ is not possible, $\mathrm{M} \in \mathcal{M}^{\mathrm{ISPM}}$ if and only if $\mathrm{M}_{11} = \min\left\{\mathrm{Q}_1^1, \mathrm{Q}_1^2\right\}$.

Let M be such that $\mathrm{M}_{23} < \min\left\{\mathrm{Q}_2^1, \mathrm{Q}_3^2\right\}$. Thus, there are at least two pairs in M that belong to one of the next categories

$$\mathrm{T}_2\mathrm{T}_1 \text{ and } \mathrm{T}_1\mathrm{T}_3 \quad \mathrm{T}_2\mathrm{T}_1 \text{ and } \mathrm{T}_3\mathrm{T}_3 \quad \mathrm{T}_2\mathrm{T}_2 \text{ and } \mathrm{T}_1\mathrm{T}_3 \quad \mathrm{T}_2\mathrm{T}_2 \text{ and } \mathrm{T}_3\mathrm{T}_3 \ .$$

Construct M′ by making one of the next rearrangements

$$\mathrm{T}_1\mathrm{T}_1 \text{ and } \mathrm{T}_2\mathrm{T}_3 \quad \mathrm{T}_3\mathrm{T}_1 \text{ and } \mathrm{T}_2\mathrm{T}_3 \quad \mathrm{T}_1\mathrm{T}_2 \text{ and } \mathrm{T}_2\mathrm{T}_3 \quad \mathrm{T}_2\mathrm{T}_3 \text{ and } \mathrm{T}_3\mathrm{T}_2 \ .$$

Note that, $\mathrm{P}(\cdot \mid \mathrm{M}\prime) \geq_{\mathrm{ISPM}} \mathrm{P}(\cdot \mid \mathrm{M})$. Since, $\mathrm{M}_{23} < \min\left\{\mathrm{Q}_2^1, \mathrm{Q}_3^2\right\}$ is not possible, $\mathrm{M} \in \mathcal{M}^{\mathrm{ISPM}}$ if and only if $\mathrm{M}_{23} = \min\left\{\mathrm{Q}_2^1, \mathrm{Q}_3^2\right\}$.

The proof that $\mathrm{M} \in \mathcal{M}^{\mathrm{ISPM}}$ if and only if $\mathrm{M}_{32} = \min\left\{\mathrm{Q}_3^1, \mathrm{Q}_2^2\right\}$ follows the same idea. ∎

Proof of Proposition 2. From Proposition 1 we know that if the distribution of types coincides in the two group, then

$$\mathrm{M}_{11} = \mathrm{Q}_1^1 = \mathrm{Q}_1^2,\ \mathrm{M}_{23} = \min\left\{\mathrm{Q}_2^1, \mathrm{Q}_3^2\right\},\ \text{and } \mathrm{M}_{32} = \min\left\{\mathrm{Q}_3^1, \mathrm{Q}_2^2\right\}$$

is payoff maximizer. Notice that, if $\alpha = 1$, then $\mathrm{T}_2\mathrm{T}_3$, $\mathrm{T}_3\mathrm{T}_2$ and $\mathrm{T}_3\mathrm{T}_3$ generate the same pair of choices $1, 1$. Thus,

$$\mathrm{M}_{11} = \mathrm{Q}_1^1 = \mathrm{Q}_1^2,\ \mathrm{M}_{23} = \mathrm{Q}_2^1 = \mathrm{Q}_2^2,\ \text{and } \mathrm{M}_{32} = \mathrm{Q}_3^1 = \mathrm{Q}_3^2$$

generates the same distribution of choices than the initial matching and it is thereby payoff equivalent. ■

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